



## Note

A note on pentavalent  $s$ -transitive graphs

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## ABSTRACT

A graph, with a group  $G$  of its automorphisms, is said to be  $(G, s)$ -transitive if  $G$  is transitive on  $s$ -arcs but not on  $(s + 1)$ -arcs of the graph. Let  $X$  be a connected  $(G, s)$ -transitive graph for some  $s \geq 1$ , and let  $G_v$  be the stabilizer of a vertex  $v \in V(X)$  in  $G$ . In this paper, we determine the structure of  $G_v$  when  $X$  has valency 5 and  $G_v$  is non-solvable. Together with the results of Zhou and Feng [J.-X. Zhou, Y.-Q. Feng, On symmetric graphs of valency five, Discrete Math. 310 (2010) 1725–1732], the structure of  $G_v$  is completely determined when  $X$  has valency 5. For valency 3 or 4, the structure of  $G_v$  is known.

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## 1. Introduction

Throughout this paper, we consider undirected finite graphs without loops or multiple edges. For a graph  $X$ , we use  $V(X)$ ,  $E(X)$  and  $\text{Aut}(X)$  to denote its vertex set, edge set, and its full automorphism group, respectively. Let  $G \leq \text{Aut}(X)$  and let  $S$  be a subset of  $V(X)$ . Denote by  $G_{(S)}$  the subgroup of  $G$  fixing  $S$  pointwise. In particular, for  $u, v, w \in V(X)$ , write  $G_u = G_{(\{u\})}$ ,  $G_{uv} = G_{(\{u, v\})}$  and  $G_{uvw} = G_{(\{u, v, w\})}$ . For  $u, v \in V(X)$ ,  $\{u, v\}$  is the edge incident to  $u$  and  $v$  in  $X$ , and  $X_1(v)$  is the neighborhood of  $v$  in  $X$ . Write  $G_v^* = G_{(\{v\} \cup X_1(v))}$  and  $G_{uv}^* = G_{(\{u, v\} \cup X_1(v) \cup X_1(u))}$ .

An  $s$ -arc in a graph  $X$  is an ordered  $(s + 1)$ -tuple  $(v_0, v_1, \dots, v_{s-1}, v_s)$  of vertices of  $X$  such that  $v_{i-1}$  is adjacent to  $v_i$  for  $1 \leq i \leq s$ , and  $v_{i-1} \neq v_{i+1}$  for  $1 \leq i \leq s - 1$ . A 1-arc is called an *arc* for short and a 0-arc is a vertex. For a subgroup  $G \leq \text{Aut}(X)$ ,  $X$  is said to be  $(G, s)$ -arc-transitive or  $(G, s)$ -regular if  $G$  is transitive or regular on the set of  $s$ -arcs in  $X$ , respectively. An  $(G, s)$ -arc-transitive graph is said to be  $(G, s)$ -transitive if the graph is not  $(G, s + 1)$ -arc-transitive. A graph  $X$  is called  $s$ -arc-transitive,  $s$ -regular or  $s$ -transitive if it is  $(\text{Aut}(X), s)$ -arc-transitive,  $(\text{Aut}(X), s)$ -regular or  $(\text{Aut}(X), s)$ -transitive, respectively. In particular,  $X$  is said to be *vertex-transitive* or *symmetric* if it is  $(\text{Aut}(X), 0)$ -arc-transitive or  $(\text{Aut}(X), 1)$ -arc-transitive, respectively.

Let  $X$  be a connected  $(G, s)$ -transitive graph for some  $s \geq 1$  and let  $G_v$  be the stabilizer of  $v \in V(X)$  in  $G$ . If  $X$  is cubic then, by Tutte [6],  $G$  is  $s$ -regular for some  $1 \leq s \leq 5$ , and by Djoković and Miller [2],  $G_v$  is isomorphic to  $\mathbb{Z}_3$ ,  $S_3$ ,  $S_3 \times \mathbb{Z}_2$ ,  $S_4$  or  $S_4 \times \mathbb{Z}_2$  for  $s = 1, 2, 3, 4$  or  $5$ , respectively. If  $X$  has valency 4, then by [1],  $G_v$  is isomorphic to a 2-group for  $s = 1$ ; by [4, Theorem 4],  $G_v$  is isomorphic to  $A_4$  or  $S_4$  for  $s = 2$  and to  $\mathbb{Z}_3 \times A_4$ ,  $\mathbb{Z}_3 \rtimes S_4$  or  $S_3 \times S_4$  for  $s = 3$ ; by [12, Theorem 1.1],  $G_v$  is isomorphic to  $\mathbb{Z}_3^2 \rtimes \text{GL}(2, 3)$  for  $s = 4$ , and to  $3^5 \rtimes \text{GL}(2, 3)$  for  $s = 7$ . Let  $X$  be of valency 5. By Weiss [7,8],  $|G_v|$  is a divisor of  $2^{17} \cdot 3^2 \cdot 5$ , and by Zhou and Feng [13], if  $G_v$  is solvable then  $G_v$  is isomorphic to  $\mathbb{Z}_5$ ,  $D_{10}$  or  $D_{20}$  for  $s = 1$ ,  $F_{20}$  or  $F_{20} \times \mathbb{Z}_2$  for  $s = 2$ ,  $F_{20} \times \mathbb{Z}_4$  for  $s = 3$ . In this paper, we determine the structure of  $G_v$  when  $G_v$  is non-solvable and therefore we obtain the following theorem.

**Theorem 1.1.** *Let  $X$  be a connected pentavalent  $(G, s)$ -transitive graph for some  $G \leq \text{Aut}(X)$  and  $s \geq 1$ . Let  $v \in V(X)$ . Then  $s \leq 5$  and one of the following holds:*

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- (1) For  $s = 1$ ,  $G_v \cong \mathbb{Z}_5, D_{10}$  or  $D_{20}$ ;
- (2) For  $s = 2$ ,  $G_v \cong F_{20}, F_{20} \times \mathbb{Z}_2, A_5$  or  $S_5$ ;
- (3) For  $s = 3$ ,  $G_v \cong F_{20} \times \mathbb{Z}_4, A_4 \times A_5, S_4 \times S_5$  or  $(A_4 \times A_5) \rtimes \mathbb{Z}_2$  with  $A_4 \rtimes \mathbb{Z}_2 = S_4$  and  $A_5 \rtimes \mathbb{Z}_2 = S_5$ ;
- (4) For  $s = 4$ ,  $G_v \cong \text{ASL}(2, 4), \text{AGL}(2, 4), \text{A}\Sigma\text{L}(2, 4)$  or  $\text{A}\Gamma\text{L}(2, 4)$ ;
- (5) For  $s = 5$ ,  $G_v \cong \mathbb{Z}_2^6 \rtimes \Gamma\text{L}(2, 4)$ .

**Remark.** For  $s = 5$ , by [12, Theorem 1.1] we have  $G_v = \langle e_0, e_1, e_2, e_3, e_4, c, f, g \rangle \cong \mathbb{Z}_2^6 \rtimes \Gamma\text{L}(2, 4)$  with the following relations:  $e_0^2 = e_1^2 = e_2^2 = e_3^2 = e_4^2 = c^3 = f^3 = g^2 = 1$ ,  $[e_0, e_1] = [e_0, e_2] = [e_1, e_2] = [e_1, e_3] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 1$ ,  $[e_0, e_3] = e_1e_2$ ,  $[e_1, e_4] = e_2e_3$ ,  $(e_0e_4)^3 = (e_0c)^3 = (e_1f)^3 = 1$ ,  $(ce_0e_4)^5 = 1$ ,  $c^{e_2} = c$ ,  $c^{e_0e_4e_0} = c^{-1}$ ,  $(cf)^{e_0} = cf$ ,  $f^{e_3} = f$ ,  $(cf)^{e_4} = cf$ ,  $[c, f] = 1$ ,  $c^g = c^{-1}$ ,  $f^g = f^{-1}$ ,  $[e_i, g] = 1$  for all  $0 \leq i \leq 4$ . By [10, Theorem 1.3],  $G_v$  has a minimal normal subgroup  $\mathbb{Z}_2^6$  such that  $G_v/\mathbb{Z}_2^6 \cong \text{A}\Gamma\text{L}(2, 4)$ , and  $G_v$  is isomorphic to a maximal subgroup (isomorphic to  $\mathbb{Z}_2^6 \rtimes ((\mathbb{Z}_3 \times A_5) \rtimes \mathbb{Z}_2)$ ) of  $\text{PSP}(4, 4) \rtimes \mathbb{Z}_2$ , where  $\mathbb{Z}_2$  is the automorphism group of  $\text{PSP}(4, 4)$  induced by field automorphism of  $\text{GF}(4)$ .

## 2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let  $X$  be a connected pentavalent  $(G, s)$ -transitive graph for some  $G \leq \text{Aut}(X)$  and  $s \geq 1$ . Let  $v \in V(X)$ . Then  $s \leq 5$  by [11, Theorem]. If  $G_v$  is solvable then, by [13, Theorem 4.1],  $G_v$  is isomorphic to  $\mathbb{Z}_5, D_{10}$  or  $D_{20}$  for  $s = 1$ ,  $F_{20}$  or  $F_{20} \times \mathbb{Z}_2$  for  $s = 2$ ,  $F_{20} \times \mathbb{Z}_4$  for  $s = 3$ . In what follows we assume that  $G_v$  is non-solvable.

Denote by  $G_v^{X_1(v)}$  the constituent of  $G_v$  on  $X_1(v)$ , that is, the permutation group induced by  $G_v$  on  $X_1(v)$ . Since  $X$  is pentavalent,  $G_v^{X_1(v)} = G_v/G_v^* \leq S_5$ . Let  $\{u, v\}$  be an edge in  $X$  and let  $S$  be a subset of  $V(X)$  such that  $u, v \in S$  and the induced subgraph  $\langle S \rangle$  of  $S$  in  $X$  is connected. Each vertex adjacent to  $S$ , but not in  $S$ , lies on an orbit of  $G_{(S)}$  with length at most 4 and then it is easy to see that  $G_{uv}$  is a  $\{2, 3\}$ -group by an inductive method on  $|S|$ . By [5, Theorem 8.5.3], each finite group of order  $p^m q^n$  with  $p, q$  primes is solvable and hence  $G_{uv}$  is solvable. In particular,  $G_v^*$  is solvable. Since  $G_v$  is non-solvable,  $G_v/G_v^*$  is non-solvable. Thus,  $G_v^{X_1(v)} = G_v/G_v^* = A_5$  or  $S_5$ . This implies that  $s \geq 2$  and  $G_v^{X_1(v) \setminus \{u\}} = A_4$  or  $S_4$ . Note that each non-trivial normal subgroup of  $A_4$  or  $S_4$  is transitive on  $X_1(v) \setminus \{u\}$ . Since  $G_u^* \triangleleft G_{uv}$ ,  $G_u^*$  acts trivially or transitively on  $X_1(v) \setminus \{u\}$ .

Assume that  $G_u^*$  acts trivially on  $X_1(v) \setminus \{u\}$ . Then  $G_u^* \leq G_v^*$  and hence  $G_u^* = G_v^*$  because the transitivity of  $G$  on  $V(X)$  implies  $|G_u^*| = |G_v^*|$ . Since  $G_v^* \triangleleft G_v$  and  $G_u^* \triangleleft G_u$ , we have  $G_v^* \triangleleft \langle G_u, G_v \rangle$ . By the transitivity of  $G$  on the set of arcs in  $X$ ,  $\langle G_u, G_v \rangle$  is transitive on the set of edges in  $X$ , and since  $G_v^*$  fixes the edge  $\{u, v\}$ , we may deduce that  $G_v^*$  fixes each edge in  $X$ , forcing that  $G_v^* = 1$ . Thus,  $G_v = A_5$  or  $S_5$ . In both cases,  $s = 2$ .

Assume that  $G_u^*$  acts transitively on  $X_1(v) \setminus \{u\}$ . Since  $s \geq 2$ , we have  $s = 3, 4$  or  $5$ .

Let  $s = 3$ . By [3, Corollary 2.3],  $|G_{uv}^*| = p^t$  for some integer  $t$  and prime  $p$ . First we claim  $|G_{uv}^*| = 1$ . Let  $(u, v, w, x)$  be a 3-arc in  $X$ . Suppose that  $G_{uv}^*$  acts non-trivially on  $X_1(w) \setminus \{v\}$ . Since  $(G_{uv}^*)^{X_1(w) \setminus \{v\}} \triangleleft (G_v^*)^{X_1(w) \setminus \{v\}} \triangleleft (G_{vw})^{X_1(w) \setminus \{v\}} = A_4$  or  $S_4$  and  $G_{uv}^*$  is a  $p$ -group, we can deduce that  $(G_{uv}^*)^{X_1(w) \setminus \{v\}}$  is a non-trivial 2-group and  $(G_{uv}^*)^{X_1(w) \setminus \{v\}} \leq A_4$ . On the other hand,  $G_{vwx}^{X_1(w) \setminus \{v, x\}} = \mathbb{Z}_3$  or  $S_3$ . Thus,  $G_{vwx}$  has an element  $g$  of order a power of 3 which induces a 3-cycle on  $X_1(w) \setminus \{v, x\}$ . Clearly,  $g$  fixes at least two vertices in  $X_1(v)$ ; one is  $w$  and since  $G$  is transitive on the set of 3-arcs in  $X$ , we may assume that the other is  $u$ . Then  $g \in G_{uvwx}$  induces a 3-cycle on  $X_1(w) \setminus \{v\}$ . Note that  $g$  normalizes  $G_{uv}^*$ , that is,  $(G_{uv}^*)^g = G_{uv}^*$ . Thus,  $G_{uv}^*$  is transitive on  $X_1(w) \setminus \{v\}$  because  $(G_{uv}^*)^{X_1(w) \setminus \{v\}} \leq A_4$  is a non-trivial 2-group. This implies that  $s \geq 4$ , a contradiction. Thus,  $G_{uv}^*$  acts trivially on  $X_1(w) \setminus \{v\}$ . In this case,  $G_{uv}^* \leq G_{vw}^*$  and since  $G$  is transitive on the set of 3-arcs in  $X$ , we have  $|G_{uv}^*| = |G_{vw}^*|$ , forcing  $G_{uv}^* = G_{vw}^*$ . By an inductive method on the distance from  $v$  to  $y$ , it is easy to see that  $G_{uv}^* = G_{yz}^*$  for any edge  $\{y, z\}$  in  $X$ . It follows that  $G_{uv}^* = 1$ , as claimed. Thus,  $G_u^*$  acts transitively and faithfully on  $X_1(v) \setminus \{u\}$ .

Set  $H = \langle G_z^* \mid z \in X_1(v) \rangle$ . Then  $H \triangleleft G_v$ . Recall that  $G_v^{X_1(v)} = A_5$  or  $S_5$ , and for each  $z \in X_1(v)$ ,  $G_z^*$  is transitive on  $X_1(v) \setminus \{z\}$ . It follows that  $H$  is transitive on  $X_1(v)$ , and hence  $H^{X_1(v)} = A_5$  or  $S_5$ . Take  $h \in H$  such that  $h$  fixes  $u$  with a 3-cycle on  $X_1(v)$  and has order a power of 3. Then  $h$  fixes some vertex  $y \in X_1(u)$  with  $y \neq v$ . Let  $x \in X_1(v)$  and let  $\alpha \in G_v^*, \beta \in G_z^*$ , where  $z$  is a fixed (but arbitrary) neighbor of  $v$ . Then  $x^{\alpha^{-1}\beta^{-1}\alpha\beta} = x^{\beta^{-1}\alpha\beta} = (x^{\beta^{-1}})^{\alpha\beta} = (x^{\beta^{-1}})^\beta = x$  and also this is true for any  $x \in X_1(z)$  because  $x^{\alpha^{-1}} \in X_1(z)$ . Thus,  $[G_v^*, G_z^*] \leq G_{vz}^* = 1$  and hence  $[G_v^*, H] = 1$ . It follows that  $H \cap G_v^* \leq Z(G_v^*)$ , the center of  $G_v^*$ . Note that  $h$  commutes with every element in  $G_v^*$ . Since  $G_u^*$  acts transitively on  $X_1(v) \setminus \{u\}$ , the arc-transitivity of  $G$  implies that  $G_v^*$  is transitive on  $X_1(u) \setminus \{v\}$ , and since  $h$  fixes  $y$ ,  $h$  fixes every vertex in  $X_1(u)$ . Thus,  $h \in G_u^*$  and  $h$  has a 3-cycle on  $X_1(v)$ . Since  $G_u^*$  acts transitively and faithfully on  $X_1(v) \setminus \{u\}$ , we have  $G_u^* = S_4$  or  $A_4$ .

By the transitivity of  $G$  on  $V(X)$ ,  $G_v^* = S_4$  or  $A_4$ . Thus,  $H \cap G_v^* \leq Z(G_v^*) = 1$ . This implies that  $H$  acts faithfully on  $X_1(v)$  and hence  $H = A_5$  or  $S_5$  because  $H^{X_1(v)} = A_5$  or  $S_5$ . Furthermore,  $G_v^*H = G_v^* \times H$ .

Now we deal with the two cases:  $G_v^* = S_4$  or  $A_4$ . If  $G_v^* = S_4$  then for each  $z \in X_1(v)$ ,  $(G_z^*)^{X_1(v) \setminus \{z\}} = S_4$ . This implies that  $H^{X_1(v)} = S_5$  because  $H = \langle G_z^* \mid z \in X_1(v) \rangle$ , and since  $H$  is faithful on  $X_1(v)$ , we have  $H = S_5$ . Similarly, if  $G_v^* = A_4$  then  $H = A_5$ . Since  $G_v/G_v^* = A_5$  or  $S_5$ , we have  $G_v = G_v^* \times H$  or  $|G_v : G_v^* \times H| = 2$ . For the former,  $G_v = S_4 \times S_5$  or  $A_4 \times A_5$ . For the latter,  $G_v^* = A_4$  and  $H = A_5$ . In this case,  $G_v = (A_4 \times A_5) \rtimes \mathbb{Z}_2$ , and to finish the proof for  $s = 3$ , it suffices to show that  $G_v = (A_4 \times A_5) \rtimes \mathbb{Z}_2$  with  $A_4 \rtimes \mathbb{Z}_2 = S_4$  and  $A_5 \rtimes \mathbb{Z}_2 = S_5$ , which is equivalent to finding an involution  $g \in G_v$  such that  $H\langle g \rangle = S_5$  and  $G_v^*\langle g \rangle = S_4$ .

Since  $|G_v : G_v^*H| = 2$ , we have  $G_v^*H = A_4 \times A_5 \triangleleft G_v$ . Recall that  $G_v^* \triangleleft G_v$  and  $H \triangleleft G_v$ . Since  $G_v^{X_1(v)} = G_v/G_v^* = S_5$ , there exists  $g_1 \in G_{uv}$  such that  $g_1$  induces a transposition on  $X_1(v)$ . Since  $G_v^* = A_4$  and  $G_v^*$  acts faithfully on  $X_1(u)$ , there is  $g_2 \in G_v^*$  such that  $g_1g_2$  induces the identity or a transposition on  $X_1(u) \setminus \{v\}$ . For the former,  $g_1g_2 \in G_u^*$  and  $g_1g_2$  induces the same transposition as  $g_1$  on  $X_1(v)$ , contrary to the fact that  $G_u^* = A_4$ , acting faithfully on  $X_1(v)$ . Set  $g = g_1g_2$ . Thus,  $g \in G_{uv}$  induces transposition on both  $X_1(u)$  and  $X_1(v)$ . Furthermore,  $g^2 \in G_{uv}^* = 1$  and hence  $g$  is an involution. It follows that  $H\langle g \rangle = S_5$  and  $G_v^*\langle g \rangle = S_4$  because  $G_v^* \triangleleft G_v$  and  $H \triangleleft G_v$ , as required.

For  $s = 4$ , by [12, Theorems 1.1 and 1.2] we have  $\mathbb{Z}_2^4 \rtimes \text{SL}(2, 4) \leq G_v \leq \mathbb{Z}_2^4 \rtimes \Gamma\text{L}(2, 4)$  and  $\mathbb{Z}_2^4 \rtimes \Gamma\text{L}(2, 4) = \langle e_0, e_1, e_2, e_3, c, f, g \rangle$ , where  $e_0^2 = e_1^2 = e_2^2 = e_3^2 = c^3 = f^3 = g^2 = 1$ ,  $[e_0, e_1] = [e_0, e_2] = [e_1, e_2] = [e_2, e_3] = 1$ ,  $[e_1, e_3] = e_2, c^{e_0e_3e_0} = c^{-1}, (e_0e_3)^3 = (e_0c)^3 = (ce_1)^3 = (ce_2)^3 = 1, (ce_0e_3)^5 = 1, [e_0, f] = 1, f^{e_1} = e_1e_1^cf, f^{e_2} = e_2e_2^cf, [e_3, f] = 1, [e_0, g] = [e_1, g] = [e_2, g] = [e_3, g] = 1, [c, f] = 1, c^g = c^{-1}, f^g = f$ . By [10, Theorem 1.3],  $\mathbb{Z}_2^4 \rtimes \Gamma\text{L}(2, 4) = \text{A}\Gamma\text{L}(2, 4)$  and hence  $\text{ASL}(2, 4) \leq G_v \leq \text{A}\Gamma\text{L}(2, 4)$ . It follows that  $G_v = \text{ASL}(2, 4), \text{AGL}(2, 4), \text{A}\Sigma\text{L}(2, 4)$  or  $\text{A}\Gamma\text{L}(2, 4)$ . For  $s = 5$ , again by [12, Theorems 1.1 and 1.2], we have  $G_v \cong \mathbb{Z}_2^6 \rtimes \Gamma\text{L}(2, 4)$ . This completes the proof of Theorem 1.1.  $\square$

### 3. Examples

Let  $X$  be a connected pentavalent  $(G, s)$ -transitive graph and let  $v \in V(X)$ . By [13, Section 3], each solvable type of  $G_v$  in Theorem 1.1 can be realized. In this section, we show that each non-solvable type of  $G_v$  in Theorem 1.1 can also be realized, that is, there exists a graph  $X$  such that  $G_v \cong A_5, S_5, A_4 \times A_5, S_4 \times S_5, (A_4 \times A_5) \rtimes \mathbb{Z}_2$  with  $A_4 \rtimes \mathbb{Z}_2 = S_4$  and  $A_5 \rtimes \mathbb{Z}_2 = S_5, \text{ASL}(2, 4), \text{AGL}(2, 4), \text{A}\Sigma\text{L}(2, 4), \text{A}\Gamma\text{L}(2, 4)$  or  $\mathbb{Z}_2^6 \rtimes \Gamma\text{L}(2, 4)$ , respectively.

**Example 3.1.** Let  $X = K_6$  be the complete graph of order 6. Then  $A = \text{Aut}(X) = S_6$ . Clearly,  $A$  has an arc-transitive subgroup  $B$  isomorphic to  $A_6$ . Thus, the vertex stabilizers  $A_v$  and  $B_v$  of  $v \in V(K_6)$  in  $A$  and  $B$  are isomorphic to  $S_5$  and  $A_5$ , respectively.

The following example is a pentavalent  $(G, 3)$ -transitive graph with  $G_v$  isomorphic to  $S_4 \times S_5, A_4 \times A_5$  or  $(A_4 \times A_5) \rtimes \mathbb{Z}_2$  with  $A_4 \rtimes \mathbb{Z}_2 = S_4$  and  $A_5 \rtimes \mathbb{Z}_2 = S_5$ .

**Example 3.2.** Let  $X = K_{5,5}$  be the complete bipartite graph of order 10 with bipartite sets  $\{1, 3, 5, 7, 9\}$  and  $\{2, 4, 6, 8, 10\}$ . Then  $A = \text{Aut}(X) \cong S_5 \text{wr} S_2$ . Clearly,  $A$  has a 3-transitive subgroup  $B$  isomorphic to  $A_5 \text{wr} S_2$ . Thus, the vertex stabilizers  $A_1$  and  $B_1$  of the vertex 1 in  $A$  and  $B$  are isomorphic to  $S_4 \times S_5$  and  $A_4 \times A_5$ , respectively. Furthermore, let  $C = \langle B, a \rangle$  with  $a = (5\ 7)(6\ 8)$ . Then  $C$  is 3-transitive and  $C_1 \cong (A_4 \times A_5) \rtimes \mathbb{Z}_2$  with  $A_4 \rtimes \mathbb{Z}_2 = S_4$  and  $A_5 \rtimes \mathbb{Z}_2 = S_5$ .

In what follows, we give two classical generalized polygon graphs (see [9,10]). The first is a 4-transitive graph.

**Example 3.3.** Let  $X$  be the classical generalized 4-gon graph related to the group  $\text{PSL}(3, 4)$ . Then  $A = \text{Aut}(X) \cong \text{Aut}(\text{PSL}(3, 4))$ . Since  $\text{Out}(\text{PSL}(3, 4)) \cong D_{12}$ , we may assume that  $\text{Aut}(\text{PSL}(3, 4)) = T \rtimes L$  with  $T = \text{Inn}(\text{PSL}(3, 4))$  and  $L = \langle a, b \mid a^2 = b^6 = 1 \rangle$ . Write  $B = \langle T, a, b^3 \rangle$ ,  $C = \langle T, b \rangle$ , and  $D = T \rtimes \langle b^3 \rangle$ . Then  $A, B, C$  and  $D$  are all 4-transitive with vertex stabilizers isomorphic to  $\text{A}\Gamma\text{L}(2, 4), \text{A}\Sigma\text{L}(2, 4), \text{AGL}(2, 4)$  and  $\text{ASL}(2, 4)$ , respectively.

At last we give a pentavalent  $(G, 5)$ -transitive graph whose vertex stabilizer  $G_v$  is isomorphic to  $\mathbb{Z}_2^6 \rtimes \Gamma\text{L}(2, 4)$ .

**Example 3.4.** Let  $X$  be the classical generalized 4-gon graph related to the group  $\text{PSp}(4, 4)$ . Then  $\text{Aut}(X) \cong \text{Aut}(\text{PSp}(4, 4))$  with a vertex stabilizer isomorphic to  $\mathbb{Z}_2^6 \rtimes \Gamma\text{L}(2, 4)$ .

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